Victorian Certificate of Education – Free Trial Examinations

MATHEMATICAL METHODS

Free Trial Written Examination 1

SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK

Abbreviations and Acronyms
- WRT – with respect to
- PDF – probability density function

Marking instructions
- The relevant line(s) of working marked give an indication of what statement should be made in order to obtain that mark, but this is subject to the marker's discretion.
- Any mark related to the method can only be awarded if the student presents a convincing (and rigorous) argument.
- The final answer mark (if any) can only be awarded if the student provides the correct answer in either simplest form or the form required.
- Consequential marks can only be obtained for marks related to the method, not for any final answer.
- If elementary mathematical steps and/or logic are broken within a solution, it must be properly justified in order to obtain full marks.

Miscellaneous notes
- Some questions may have multiple methods/solutions, including some that are beyond the scope of the course. The solutions provided are the ones that were intended by the examination authors.

For any questions, concerns and/or possible errors, please contact us via the contact form on our website.
Version 2 (February 2019)

Question 1a (2 marks)

Mark	Criteria
1	Applies quotient rule, or equivalent
2	Provides correct answer

$$\frac{dy}{dx}=\frac{\frac{d}{dx}[\sin(x)]\cdot 3x^2-\frac{d}{dx}\left[3x^2\right]\sin(x)}{\left(3x^2\right)^2}$$

$$=\frac{3x^2\cos(x)-6x\sin(x)}{9x^4}$$ **Mark 1**

$$=\frac{x\cos(x)-2\sin(x)}{3x^3}$$ **Mark 2**

Question 1b (2 marks)

Mark	Criteria
1	Differentiates f WRT x using the chain rule
2	Provides correct answer

$$f'(x)=4\frac{d}{dx}\left[\sqrt{x}\right]e^{\sqrt{x}}$$

$$=\frac{2}{\sqrt{x}}e^{\sqrt{x}}$$ **Mark 1**

$$f'(4)=e^2$$ **Mark 2**

Question 2a.i (1 mark)

Mark	Criteria
1	Provides a correct method

$1+\frac{2}{x-2}=\frac{x-2+2}{x-2}=\frac{x}{x-2}$, as required. **Mark 1**

Question 2a.ii (1 mark)

Mark	Criteria
1	Provides correct answer

$$\int g(x)\,dx=\int\left(1+\frac{2}{x-2}\right)dx$$

$$=x+2\log_e(x-2)\quad[\text{since } x-2>0]$$ **Mark 1**

Question 2b (2 marks)

Mark	Criteria
1	Antidifferentiates f' WRT x
2	Provides correct answer

$$f(x)=\int\left(\pi\cos(\pi x)+2x^{-1/2}\right)dx$$

$$=\sin(\pi x)+4x^{1/2}+c\quad[c\in\mathbb{R}]$$ **Mark 1**

Since $f\left(\frac{1}{4}\right)=\frac{1}{\sqrt{2}}$, we have $\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}+2+c$, and so $c=-2$.

Therefore, $f(x)=\sin(\pi x)+4\sqrt{x}-2$. **Mark 2**

Question 3a (2 marks)

Mark	Criteria
1	Obtains correct reference angle, or equivalent
2	Provides correct answer

$\sin(\pi x)=\frac{\sqrt{3}}{2}$, where $-2\pi<\pi x<\pi$

$$\pi x=\frac{-5\pi}{3},\frac{-4\pi}{3},\frac{\pi}{3},\frac{2\pi}{3}$$ **Mark 1**

$$x=\frac{-5}{3},\frac{-4}{3},\frac{1}{3},\frac{2}{3}$$ **Mark 2**

Question 3b (2 marks)

Mark	Criteria
1	Removes logarithms from equation
2	Provides correct answer with justification

$\log_e\left[\dfrac{(a+3)^2}{b^2}\right]=2$

$\dfrac{(a+3)^2}{b^2}=e^2$ **Mark 1**

$(a+3)^2=e^2b^2$

$a+3=-eb$ $[a+3\neq +eb$ since we require $a+3>0$ with $b<0]$

Therefore, $a=-3-eb$. **Mark 2**

Question 4a (1 mark)

Mark	Criteria
1	Provides correct answer

$L\sim \mathrm{N}\left(100, 8^2\right)$

$$\begin{aligned}\Pr(L>108)&=\Pr\left(Z>\frac{108-100}{8}\right)\\&=\Pr(Z>1)\\&=0.16\quad(2\text{DP})\end{aligned}$$ **Mark 1**

Question 4b (2 marks)

Mark	Criteria
1	Applies conditional probability definition and utilises symmetry, either algebraically or graphically
2	Provides correct answer

$$\begin{aligned}\Pr(L>92\mid L<100)&=\frac{\Pr(92<L<100)}{\Pr(L<100)}\\&=\frac{\Pr(-1<Z<0)}{\Pr(Z<0)}\\&=\frac{0.5-0.16}{0.5}\end{aligned}$$ **Mark 1**

$$=0.68\quad(2\text{DP})$$ **Mark 2**

Question 4c (2 marks)

Mark	Criteria
1	Writes down an inequation involving n, or equivalent
2	Provides correct answer

$\Pr(L<100)=0.5,$ so we have

$\sqrt{\dfrac{1/2\times 1/2}{n}}\leq\dfrac{1}{48}$ **Mark 1**

$\dfrac{1}{2\sqrt{n}}\leq\dfrac{1}{48}$

$\sqrt{n}\geq 24$

Therefore, the smallest value of n is 576. **Mark 2**

Question 5a (2 marks)

Mark	Criteria
1	Finds zeros of f'
2	Provides correct answer

Let $f'(x) = \frac{1}{4}\left(3x^2 - 3\right) = 0.$

$x^2 - 1 = 0$

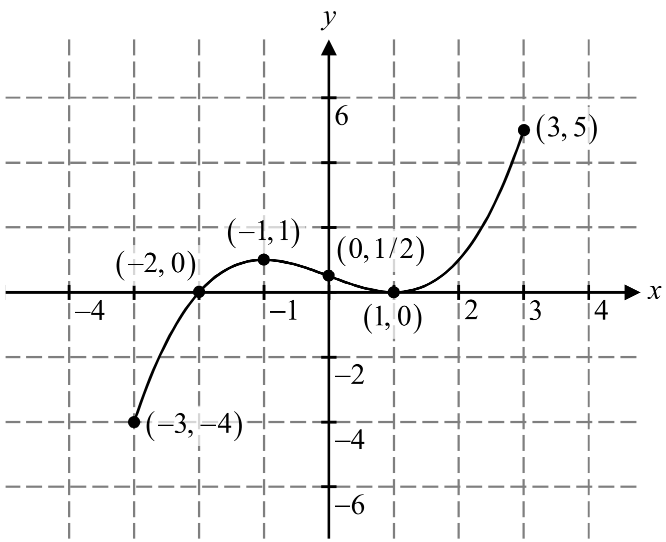
$x = -1, 1$ **Mark 1**

$f(-1) = 1$ and $f(1) = 0.$

The stationary points of f are $(-1, 1)$ and $(1, 0)$. **Mark 2**

Question 5b (2 marks)

Mark	Criteria
1	Labels all points correctly
2	Sketches correct graph shape



Question 5c (1 mark)

Mark	Criteria
1	Provides correct answer

From the graph, $\bar{f} = \frac{1}{2}$. **Mark 1**

Question 6a (3 marks)

Mark	Criteria
1	Finds expression for n in terms of m
2	Expands resulting quadratic for Var(X)
3	Provides correct answer

Since $0.1 + m + n = 1$, we have $n = 0.9 - m$. **Mark 1**

$\text{Var}(X) = \text{E}\left(X^2\right) - [\text{E}(X)]^2$

$= m + 4n - (m + 2n)^2$

$= -3m + 3.6 - (1.8 - m)^2$

$= -3m + 3.6 - \left(3.24 - 3.6m + m^2\right)$ **Mark 2**

$= -m^2 + 0.6m + 0.36$

Hence, $a = -1, b = 0.6$ and $c = 0.36$. **Mark 3**

Question 6b (1 mark)

Mark	Criteria
1	Provides a correct method

Provided m is suitable, maximum variance occurs when $m = \frac{-0.6}{2 \times (-1)} = 0.3$,

since the $\text{Var}(X)$ is given by a 'negative' quadratic, as required. **Mark 1**

Question 6c (2 marks)

Mark	Criteria
1	Writes down the possible event combinations, and their associated probabilities, or equivalent
2	Provides correct answer

$\Pr(E) = \Pr(1,2) + \Pr(2,1) + \Pr(2,2)$

$= 0.3\times 0.6 + 0.6\times 0.3 + 0.6\times 0.6$ **Mark 1**

$= 0.18 + 0.18 + 0.36$

$= 0.72$ **Mark 2**

Question 7a.i (1 mark)

Mark	Criteria
1	Provides a correct method

$h(x) = \sqrt{x^2 - 2x + 5}$

$= \sqrt{(x-1)^2 - 1 + 5}$ **Mark 1**

$= \sqrt{(x-1)^2 + 4}$, as required.

Question 7a.ii (2 marks)

Mark	Criteria
1	Provides correct domain
2	Provides correct range

$\text{domain}(h) = \text{domain}(g) = (-\infty, 1]$ **Mark 1**

Since $\text{range}(g) = [-1, \infty)$ and f is strictly increasing, we have

$[-1,\infty) \xrightarrow{f} [2,\infty)$, and so $\text{range}(h) = [2, \infty)$. **Mark 2**

Question 7b (2 marks)

Mark	Criteria
1	Provides correct rule of inverse function of h
2	Provides correct domain and range

Let $y = h^{-1}(x)$.

$x^2 = (y-1)^2 + 4$

$y - 1 = \pm\sqrt{x^2 - 4}$

However, since we require $h^{-1}(x) \le 1$, we have $h^{-1}(x) = 1 - \sqrt{x^2 - 4}$. **Mark 1**

$\text{domain}\left(h^{-1}\right) = [2, \infty)$ and $\text{range}\left(h^{-1}\right) = (-\infty, 1]$. **Mark 2**

Question 8a.i (2 marks)

Mark	Criteria
1	Provides one correct transformation
2	Provides correct answer

- Dilation by factor $\frac{1}{a}$ from the x-axis **Mark 1**
- Dilation by factor $\frac{1}{a}$ from the y-axis **Mark 2**

Note: in any order

Question 8a.ii (1 mark)

Mark	Criteria
1	Provides a correct method

Method 1:

The first positive x-axis intercept of the graph of $y = x\cos(x)$ is $\left(\frac{\pi}{2}, 0\right)$,

and so applying the transofmrations from **part a.i**, we have

$(b, 0) = \left(\frac{\pi}{2} \times \frac{1}{a},\ 0 \times \frac{1}{a}\right) = \left(\frac{\pi}{2a}, 0\right)$. **Mark 1**

Thus, $b = \frac{\pi}{2a}$, as required.

Method 2:

Let $x\cos(ax) = 0$, where $x > 0$.

$\cos(ax) = 0$

$ab = \frac{\pi}{2}$ [for first positive x-axis intercept] **Mark 1**

Thus, $b = \frac{\pi}{2a}$, as required.

Question 8b (4 marks)

Mark	Criteria
1	Differentiates $x\sin(ax)$ WRT x using the product rule.
2	Forms an equation involving an indefinite integral and finds an antiderivative of $x\cos(ax)$ WRT x
3	Substitutes $a = \pi/(2b)$
4	Provides correct answer

$\frac{d}{dx}[x\sin(ax)] = \frac{d}{dx}[x]\sin(ax) + x\frac{d}{dx}[\sin(ax)]$

$= \sin(ax) + ax\cos(ax)$ **Mark 1**

Since f is a PDF, we have

$1 = \int_0^b x\cos(ax)\,dx$

$= \frac{1}{a}[x\sin(ax)]_0^b - \frac{1}{a}\int_0^b \sin(ax)\,dx$ [using above result]

$= \frac{1}{a}[x\sin(ax)]_0^b + \frac{1}{a}\left[\frac{1}{a}\cos(ax)\right]_0^b$ **Mark 2**

Substituting $a = \frac{\pi}{2b}$ gives

$\left[\frac{2bx}{\pi}\sin\left(\frac{\pi x}{2b}\right) + \frac{4b^2}{\pi^2}\cos\left(\frac{\pi x}{2b}\right)\right]_0^b = 1$ **Mark 3**

$\frac{2b^2}{\pi} + 0 - 0 - \frac{4b^2}{\pi^2} = 1$

$b^2(2\pi - 4) = \pi^2$

$b = \frac{\pi}{\sqrt{2\pi - 4}}$ $[b > 0]$ **Mark 4**

END OF SUGGESTED SOLUTIONS AND MARKING GUIDE BOOK