VCAA 2019 Mathematical Methods
Examination 1 Provisional Solutions

TWM Publications

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Question 1a.i (1 mark)

If $f : \left(\frac{1}{3},\ \infty\right) \to \mathbb{R},\ f(x) = \frac{1}{3x-1} = (3x-1)^{-1}$, then by the chain rule,

$$f'(x) = -3(3x-1)^{-2} = \frac{-3}{(3x-1)^2}$$

Question 1a.ii (1 mark)

Noting that $3x - 1 > 0$, an antiderivative of f is

$$\int f(x)\,\mathrm{d}x = \frac{1}{3}\log_e(3x-1), \quad (c = 0).$$

Question 1b (2 marks)

If $g : \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ g(x) = \frac{\sin(\pi x)}{x+1}$, then by the quotient rule,

$$g'(x) = \frac{\pi\cos(\pi x)(x+1) - \sin(\pi x)}{(x+1)^2} \implies g'(1) = \frac{-\pi}{2}.$$

Question 2a (2 marks)

If $f : \mathbb{R} \setminus \left\{\frac{1}{3}\right\} \to \mathbb{R},\ f(x) = \frac{1}{3x-1}$, then

$$x = \frac{1}{3f^{-1}(x) - 1} \implies 3f^{-1}(x) - 1 = \frac{1}{x} \implies f^{-1}(x) = \frac{1}{3x} + \frac{1}{3}.$$

Question 2b (1 mark)

We have $\mathrm{dom}(f^{-1}) = \mathrm{ran}(f) = \mathbb{R} \setminus \{0\}$.

Question 2c (1 mark)

The graph of $g = f^{-1}$ is obtained from the graph of f by a translation of 1/3 units in the negative x-direction and a translation of 1/3 units in the positive y-direction. Thus,

$$c = \frac{-1}{3} \quad \text{and} \quad d = \frac{1}{3}.$$

Question 3a (2 marks)

There are two unbiased coins and one biased coin (1/3 heads). Let E denote tossing a head from a randomly selected coin from her pocket. Then,

$$\Pr(E) = \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) + \left(\frac{1}{3} \times \frac{1}{3}\right) = \frac{4}{9}.$$

Question 3b (1 mark)

Let U denote choosing an unbiased coin. Then,

$$\Pr(U \,|\, E) = \frac{\Pr(U \cap E)}{\Pr(E)} = \frac{2/3 \times 1/2}{4/9} = \frac{3}{4}.$$

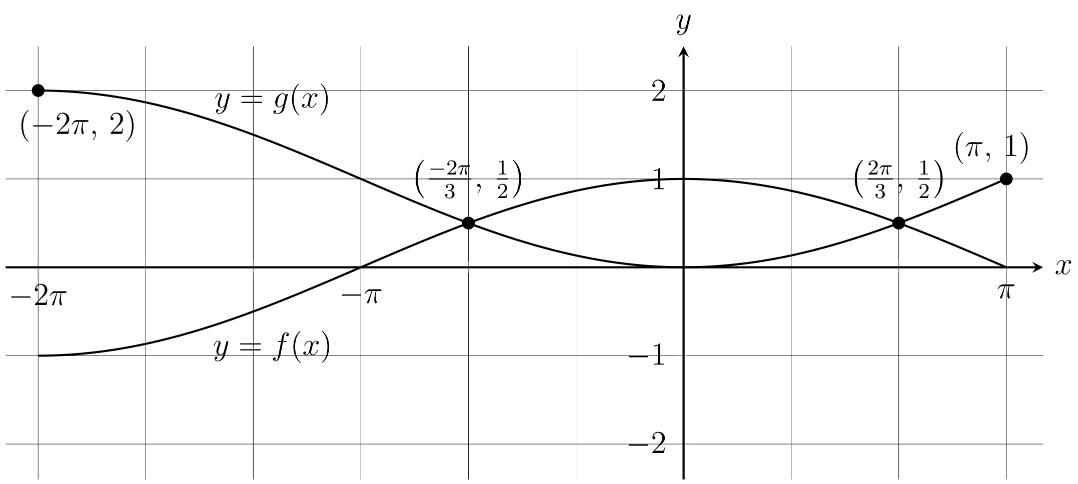
Question 4a (2 marks)

Given $x \in [-2\pi,\, \pi]$, we have

$$1 - \cos\left(\frac{x}{2}\right) = \cos\left(\frac{x}{2}\right) \implies \cos\left(\frac{x}{2}\right) = \frac{1}{2} \implies x = \frac{\pm 2\pi}{3}.$$

Question 4b (2 marks)

Where $f : [-2\pi, \pi] \to \mathbb{R},\ f(x) = \cos\left(\frac{x}{2}\right)$, the graphs of $y = f(x)$ and $y = g(x) = 1 - f(x)$ are shown below.



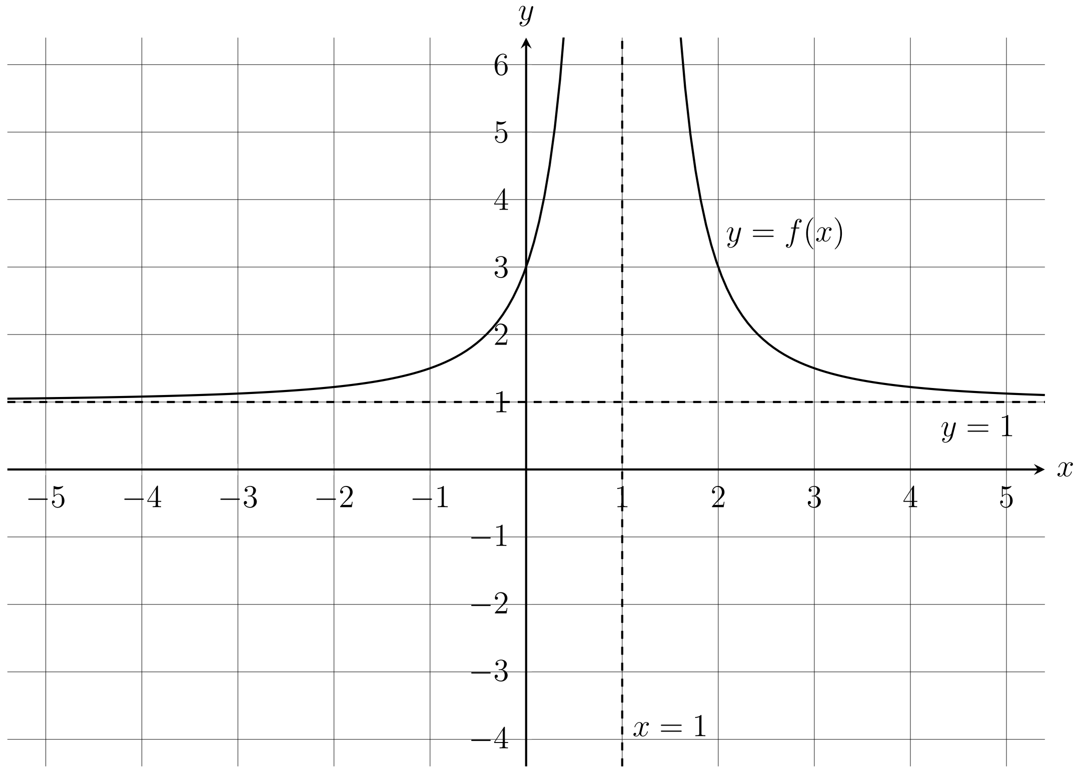
Question 5a.i (1 mark)

Here, $f : \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f(x) = \dfrac{2}{(x-1)^2} + 1$, and so

$$f(-1) = \frac{3}{2}.$$

Question 5a.ii (2 marks)

Shown below is the graph of $y = f(x)$.



Question 5b (2 marks)

The area bounded by the graph of f, the x-axis, the line $x = 0$ and the line $x = -1$ is

$$A = \int_{-1}^{0} \left(\frac{2}{(x-1)^2} + 1 \right) \mathrm{d}x = \left[x - \frac{2}{x-1} \right]_{-1}^{0} = 2 \text{ units}^2.$$

Question 6a (1 mark)

Of the 41 pegs produced, eight are faulty, so $\widehat{p}_0 = \dfrac{8}{41}$.

Question 6b (2 marks)

Let $X \sim \text{Bi}(12,\ 1/6)$ denote the number of faulty pegs in a randomly selected box. Then,

$$\Pr\left(\widehat{P} < \frac{1}{6}\right) = \Pr(X \leq 1) = \binom{12}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{12} + \binom{12}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{11} = \frac{17}{6}\left(\frac{5}{6}\right)^{11}.$$

Question 7a (1 mark)

The distance PB is given by $d(PB) = \sqrt{1-x^2}$.

Question 7b (3 marks)

The area of the triangle ABP is given by

$$A(x) = \frac{1}{2}(x+1)\sqrt{1-x^2},$$

and so to maximise the area, we compute

$$A'(x) = \frac{1}{2}\sqrt{1-x^2} - \frac{1}{2}(x+1)\frac{-2x}{2\sqrt{1-x^2}} = \frac{1-x-2x^2}{2\sqrt{1-x^2}}.$$

Hence, we have

$$A'(x) = 0 \implies 2x^2 + x - 1 = 0 \implies (x+1)(2x-1) = 0 \implies x = \frac{1}{2}$$

for maximum area. Thus, the maximum area of the triangle is

$$A_{\text{max}} = A\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{8} \text{ units}^2.$$

Question 8a (1 mark)

The rule of f has the form

$$f(x) = ax^2(x+1)(x-1), \quad a \in \mathbb{R},$$

and since $f\left(1/\sqrt{2}\right) = 1$, we have $a = -4$. Thus,

$$f(x) = -4x^2(x+1)(x-1).$$

Question 8b (1 mark)

Here, $h : D \to \mathbb{R},\ h(x) = \log_e\big(g(x)\big) - \log_e(x^3 + x^2)$, where D is the maximal domain of h and g has the same rule as f. We require that

$$g(x) > 0 \quad \text{and} \quad x^3 + x^2 = x^2(x+1) > 0 \implies D = (-1,\ 1) \setminus \{0\}.$$

Question 8c (2 marks)

First, we have

$$h(x) = \log_e\big(-4x^2(x+1)(x-1)\big) - \log_e\big(x^2(x+1)\big) = \log_e(4-4x), \quad x \in (-1,\ 1) \setminus \{0\}.$$

Since h is strictly decreasing, the range of h is given by

$$\text{ran}(h) = \big(h(1^-),\ h(-1^+)\big) \setminus \{\log_e(4-0)\} = (-\infty,\ \log_e(8)) \setminus \{\log_e(4)\}.$$

Question 9a (1 mark)

Here, we have $f : \mathbb{R} \to \mathbb{R},\ f(x) = 3 + 2x - x^2$ and $g : \mathbb{R} \to \mathbb{R},\ g(x) = e^x$. Thus,

$$g(f(x)) = e^{3+2x-x^2}.$$

Question 9b (2 marks)

By the chain rule, we have

$$\frac{d}{dx}\big[g(f(x))\big] = (2-2x)e^{3+2x-x^2} < 0 \implies 2 - 2x < 0 \implies x \in (1,\, \infty),$$

noting that $g(f(x)) > 0$ for all $x \in \mathbb{R}$.

Question 9c (1 mark)

We have $f(g(x)) = 3 + 2e^x - e^{2x}$.

Question 9d (2 marks)

Here, $f(g(x)) = 0$ gives

$$(3 - e^x)(e^x + 1) = 0 \implies e^x = 3 \implies x = \log_e(3),$$

noting that $e^x > 0$ for all $x \in \mathbb{R}$.

Question 9e (2 marks)

By the chain rule,

$$\frac{d}{dx}\big[f(g(x))\big] = 2e^x - 2e^{2x} = 0 \implies 2 - 2e^x = 0 \implies x = 0,$$

again, noting that $e^x \neq 0$ for all $x \in \mathbb{R}$. Hence, the stationary point is at $(0,\, 4)$.

Question 9f (1 mark)

Using *Q9b*, *Q9d* and *Q9e*, one can produce a rough sketch for the graphs of $y = f(g(x))$ and $y = g(f(x))$. Clearly, there is one solution for x.